PATH INTEGRAL QUANTIZATION OF LANDAU-GINZBURG THEORY

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Abstract

Hamilton-Jacobi approach for a constrained system is discussed. The equation of motion for a singular systems are obtained as total differential equations in many variables. The integrability conditions are investigated without using any gauge fixing condition. The path integral quantization for systems with finite degrees of freedom is applied to the field theories with constraints. The Landau-Ginzburg theory is investigated in details.

1 Introduction

The quantization of constrained Hamiltonian systems can be achieved by means of operator methods [1,2] or by path integral quantization [3-5].

The quantization of a massive spin one particle became the center of interest of physicists especially after the pioneering work of Faddeev [3] who introduced the path integral quantization of singular theories which possess first-class constraints in canonical gauge. Since first class constraints are generators of gauge transformations, this will lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional arbitrariness, one has to impose external gauge constraints for each first-class constraint.

The gauge fixing is not always an easy task if Dirac's method is used. The gauge fixing is not necessary to analyze singular systems [6] if the canonical method or Hamilton-Jacobi approach [7,8] is used.

The aim of this paper is to treat the Landau-Ginzburg theory that gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics as constrained system. The path integral quantization is obtained using canonical method.

2 Hamilton-Jacobi Formalism Of Constrained Systems

In this section, we study the constrained systems by using the canonical method [7,8] and demonstrate the fact that the gauge fixing problem is solved naturally. The starting point of this method is to consider the lagrangian $L \equiv L(q_i, \dot{q}_i, \tau),\ i = 1, 2, \ldots, n$ with Hessian matrix

$$A_{ij} = \frac{\partial^2 L(q_i, \dot{q}_i, \tau)}{\partial \dot{q}_i\, \partial \dot{q}_j}, \qquad i, j = 1, 2, \ldots, n, \tag{1}$$

of rank $(n-r)$, $r<n$. Then r momenta are dependent. The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, 2, \ldots, n-r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-r+1, \ldots, n. \tag{3}$$

enable us to solve Eq. (2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, p_a, \dot{q}_\mu; \tau) \equiv \omega_a. \tag{4}$$

Substituting Eq. (4), into Eq. (3), we get

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a=\omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \tag{5}$$

Relations (5) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the lagrangian.
The canonical Hamiltonian H_0 is defined as

$$H_0 = -L(q_i, \dot{q}_\mu, \dot{q}_a \equiv \omega_a; \tau) + p_a \dot{q}_a + p_\mu \dot{q}_\mu \big|_{p_\mu=-H_\mu}. \tag{6}$$

The set of Hamilton-Jacobi Partial Differential Equations is expressed as

$$H'_\alpha\left(\tau,\ q_\mu,\ q_a,\ P_i = \frac{\partial S}{\partial q_i},\ P_0 = \frac{\partial S}{\partial \tau}\right) = 0,\ \alpha = 0, n-p+1, \ldots, n, \tag{7}$$

where

$$H'_\alpha = p_\alpha + H_\alpha\ . \tag{8}$$

The equations of motion are obtained as total differential equation in many variables as follows:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r}\, dt_\alpha, \qquad r = 0, 1, \ldots, n\,, \tag{9}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha, \qquad a = 1, \ldots, n-p, \tag{10}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}\, dt_\alpha, \qquad \alpha = 0, n-p+1, \ldots, n, \tag{11}$$

$$dz = \Big(- H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \Big) dt_\alpha. \tag{12}$$

where $Z = S(t_\alpha, q_a)$. The set of equations (9-12) are integrable if

$$dH'_\alpha = 0, \qquad \alpha = 0, n-p+1, \ldots, n. \tag{13}$$

If conditions (13) are not satisfied identically, one may consider them as new constraint and a gain test the integrability conditions, then repeating this procedure, a set of conditions may be obtained.
In this case of integrable system, the path integral representation may be written as [9-13].

$$\langle Out \mid S \mid In \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp\left[i \int_{t_\alpha}^{t'_\alpha} \Big(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \Big) dt_\alpha \right]. \tag{14}$$

One should notice that the integral (14) is an integration over the canonical phase space coordinates q_a, p_a.

3 The Landau-Ginzburg theory

The Landau-Ginzburg theory that gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics is described by the lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* D^\mu \varphi - k \varphi^* \varphi - \frac{1}{4} \lambda (\varphi^* \varphi)^2, \tag{15}$$

where

$$D_\mu \varphi = \partial_\mu \varphi - ie A_\mu \varphi. \tag{16}$$

In the landau-Ginzburg theory φ describes the cooper pairs. In usual quantum electrodynamics, we would put $k = m^2$, where m is the effective mass of electron.
The Lagrangian function (15) is singular, since the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{17}$$

is three.
The canonical momenta are defined as

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = -F^{0i}, \tag{18}$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0, \tag{19}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = (D_0\varphi)^* = \dot{\varphi}^* + ieA_0\varphi^*, \tag{20}$$

$$p_{\varphi^*} = \frac{\partial L}{\partial \dot{\varphi}^*} = (D_0\varphi) = \dot{\varphi} - i\, e\, A_0\, \varphi, \tag{21}$$

From Eqs. (18), (20) and (21), the velocities $\dot{A}_i, \dot{\varphi}^*$ and $\dot{\varphi}$ can be expressed in terms of momenta π_i, p_φ and p_{φ^*} respectively as

$$\dot{A}_i = -\pi_i - \partial_i A_0, \tag{22}$$

$$\dot{\varphi}^* = p_\varphi - ieA_0\varphi^*, \tag{23}$$

$$\dot{\varphi} = p_{\varphi^*} + ieA_0\varphi. \tag{24}$$

The canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 = \frac{1}{4}\, F^{ij}F_{ij} - \frac{1}{2}\, \pi_i\pi^i + \pi^i\, \partial_i A_0 + p_{\varphi^*}p_\varphi + ieA_0\varphi p_\varphi \\ - ieA_0\varphi^* p_{\varphi^*} - (D_i\varphi)^*(D^i\varphi) + k\varphi^*\varphi + \frac{1}{4}\lambda(\varphi^*\varphi)^2. \end{aligned} \tag{25}$$

Making use of (7) and (8), we find for the set of HJPDE

$$H_0' = \pi_4 + H_0, \tag{26}$$

$$H' = \pi_0 + H = \pi_0 = 0, \tag{27}$$

Therefor, the total differential equations for the characteristic (9-11) obtained as

$$\begin{aligned} dA^i &= \frac{\partial H_0'}{\partial \pi_i}\, dt + \frac{\partial H'}{\partial \pi_i}\, dA^0, \\ &= -(\pi^i + \partial_i A_0)\, dt, \end{aligned} \tag{28}$$

$$dA^0 = \frac{\partial H_0'}{\partial \pi_0}\, dt + \frac{\partial H'}{\partial \pi_0}\, dA^0 = dA^0, \tag{29}$$

$$\begin{aligned} d\varphi &= \frac{\partial H_0'}{\partial p_\varphi}\, dt + \frac{\partial H'}{\partial p_\varphi}\, dA^0, \\ &= (p_{\varphi^*} + ieA_0\varphi)\, dt, \end{aligned} \tag{30}$$

$$\begin{aligned} d\varphi^* &= \frac{\partial H_0'}{\partial p_{\varphi^*}}\, dt + \frac{\partial H'}{\partial p_{\varphi^*}}\, dA^0, \\ &= (p_\varphi - ieA_0\varphi^*)\, dt, \end{aligned} \tag{31}$$

$$\begin{aligned} d\pi^i &= -\frac{\partial H_0'}{\partial A_i}\, dt - \frac{\partial H'}{\partial A_i}\, dA^0, \\ &= [\partial_l F^{li} + ie(\varphi^* \partial^i \varphi + \varphi\, \partial_i \varphi^*) + 2e^2 A^i \varphi\varphi^*]\, dt, \end{aligned} \tag{32}$$

$$\begin{aligned} d\pi^0 &= -\frac{\partial H_0'}{\partial A_0}\, dt - \frac{\partial H'}{\partial A_0}\, dA^0, \\ &= [\partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\, p_\varphi]\, dt, \end{aligned} \tag{33}$$

$$\begin{aligned} dp_\varphi &= -\frac{\partial H_0'}{\partial \varphi}\, dt - \frac{\partial H'}{\partial \varphi}\, dA^0, \\ &= [(\overrightarrow{D} \cdot \overrightarrow{D}\varphi)^* - k\varphi^* - \frac{1}{2}\lambda\varphi\varphi^{*2} - ieA_0 p_\varphi]\, dt, \end{aligned} \tag{34}$$

and

$$\begin{aligned} dp_{\varphi^*} &= -\frac{\partial H_0'}{\partial \varphi^*}\, dt - \frac{\partial H'}{\partial \varphi^*}\, dA^0, \\ &= [(\overrightarrow{D} \cdot \overrightarrow{D}\varphi) - k\varphi - \frac{1}{2}\lambda\varphi^*\varphi^2 + ieA_0 p_{\varphi^*}]\, dt. \end{aligned} \tag{35}$$

The integrability condition ($dH'_\alpha = 0$) implies that the variation of the constraint H' should be identically zero, that is

$$dH' = d\pi_0 = 0, \tag{36}$$

which leads to a new constraint

$$H'' = \partial_i \pi^i + ie\varphi^* p_{\varphi^*} - ie\varphi\, p_\varphi = 0. \tag{37}$$

Taking the total differential of H'', we have

$$dH'' = \partial_i d\pi^i + iep_{\varphi^*} d\varphi^* + ie\varphi^* dp_{\varphi^*} - ie\varphi\, dp_\varphi - iep_\varphi\, d\varphi = 0. \tag{38}$$

Then the set of equation (28-35) is integrable. Therefore, the canonical phase space coordinates (φ, p_φ) and $(\varphi^*, p_{\varphi^*})$ are obtained in terms of parameters (t, A^0).

Making use of Eq.(12) and (25-27), we obtain the canonical action integral as

$$Z = \int d^4x(-\frac{1}{4}F^{ij}F_{ij} - \frac{1}{2}\pi_i\pi^i + p_\varphi p_{\varphi^*} + \overrightarrow{D}\varphi^* \cdot \overrightarrow{D}\varphi + k|\varphi|^2 + \frac{1}{4}\lambda(\varphi^*\varphi)^2), \tag{39}$$

where

$$\overrightarrow{D} = \overrightarrow{\nabla} + ie\overrightarrow{A}. \tag{40}$$

Now the path integral representation (14) is given by

$$\langle out|S|In\rangle = \int \prod_i dA^i\, d\pi^i\, d\varphi\, dp_\varphi\, d\varphi^*\, dp_{\varphi^*}\, exp\Bigg[i\Bigg\{\int d^4x$$
$$\Bigg(-\frac{1}{2}\,\pi_i\pi^i - \frac{1}{4}\,F^{ij}F_{ij} + p_\varphi p_{\varphi^*} + (D_i\varphi)^*(D_i\varphi) - k\varphi^*\varphi - \frac{1}{4}\lambda(\varphi^*\varphi)^2\Bigg)\Bigg\}\Bigg]. \tag{41}$$

4 Conclusion

In this paper the Landau-Ginzburg theory that gives an effective description of phenomenon precisely coincides with scalar quantum electrodynamics has been quantized by constructing a path integral quantization

within the canonical method to constrained system. The equations of motion are obtained as total differential equations in many variables. All the constraints coming from the Hamiltonian procedure and the integrability conditions have been derived. The path integral quantization is performed using the action given by canonical method, and the integration is taken over the canonical phase space.

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